Canonical rings of stacky surfaces

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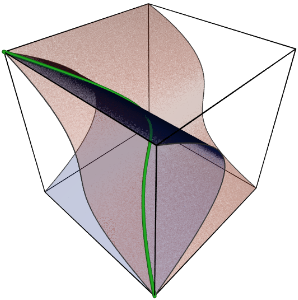
Algebraic Shapes



In Algebraic Geometry, we study manifolds that can be described by polynomials.

Example (Twisted cubic curve)

$$Y = \{(t, t^2, t^3) : t \in \mathbb{C}\}^a$$

[a] By Claudio Rocchini - Own work, CC BY 3.0

Prefer to work in Projective Space

$$\mathbb{P}(V) = {}^{V\setminus\{0\}}/_{\mathbb{C}^\times}, \quad v \sim \lambda v$$

In $\mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$, the twisted cubic is

$$X = \{[s^3 : s^2 t : st^2 : t^3] : s, t \in \mathbb{C}\} \subseteq \mathbb{P}^3$$

Same shape, different guise

A map $X \to \mathbb{P}^n$ is given by $p \mapsto [s_0(p) : s_1(p) : \cdots : s_n(p)]$.

Example (Twisted cubic)

The map $\varphi : \mathbb{P}^1 \to \mathbb{P}^3 : [s : t] \to [s^3 : s^2t : st^2 : t^3]$ induces an isomorphism $\mathbb{P}^1 \simeq X$.

Composing with linear maps, we get a complete linear system of such maps. Write $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ for the complete linear system of homogeneous degree d polynomials.

Example

$$\varphi \longleftrightarrow H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(3))$$

The canonical map

The holomorphic differentials Ω_X give rise to a canonical map.

Example (Fermat quartic)

$$X = \{[x : y : z] \in \mathbb{P}^2 \mid x^4 + y^4 + z^4 = 0\}$$

Then from

$$x^3\ dx + y^3\ dy + z^3\ dz \Rightarrow \frac{z\ dx - x\ dz}{y^3} = \frac{y\ dz - z\ dy}{x^3} =: \omega$$

get the complete linear system

$$H^0(X, \Omega_X) = \langle x\omega, y\omega, z\omega\rangle \simeq H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)),$$

showing that X is canonically embedded in $\mathbb{P}^2$.

The genus of a curve X is $g = \dim H^0(X, \Omega_X)$.

The one ring

The canonical ring of a variety X is

$$R(X) := \bigoplus_{d=0}^{\infty} H^0(X, \omega_X^{\otimes d})$$

where $\omega_X = \det \Omega_X$. [1]

Reasons for interest -

- X a curve, $g \geq 2 \implies \operatorname{Proj} R(X) \simeq X$. (Petri, 1923)
- X a variety $\implies \kappa(X) = \dim \operatorname{Proj} R(X)$.
- Models of X in weighted projective space.

[1] By Peter J. Yost - Own work, CC BY-SA 4.0

Rings of power(s)

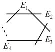
More generally, for linear systems on X can consider

$$R(X, \mathscr{F}) := \bigoplus_{d=0}^{\infty} H^0(X, \mathscr{F}^{\otimes d})$$

Example (Twisted cubic)

If $X = \mathbb{P}^1$ and $\mathscr{F} = \mathscr{O}_{\mathbb{P}^1}(3)$, then

$$R(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2)) = \bigoplus_{d=0}^{\infty} H^0(X, \mathscr{O}_{\mathbb{P}^1}(3d)) = \mathbb{C}[s^3, s^2t, st^2, t^3]$$

Important special case - Δ is NCD 

and $\mathscr{F} = \omega_X(\log \Delta)$.

Modular forms

- $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ discrete and torsion-free, acting on $\mathcal{H}$.
- $Y = \Gamma \backslash \mathcal{H}$ is an algebraic curve, $X = Y \cup \Delta$ compact.
- The complete linear system

$$H^0(X, \Omega_X^{\otimes d}) = S_{2d}(\Gamma)$$

is the space of cusp forms of weight $k = 2d$.
- The complete linear system

$$H^0(X, \Omega_X(\log \Delta)^{\otimes d}) = M_{2d}(\Gamma)$$

is the space of modular forms of weight $k = 2d$.
- Can recover the modular curve X from the modular forms.

What if Γ has torsion?

Stacky curves

- If Γ has torsion, $\mathscr{X} = \Gamma\backslash\mathcal{H} \cup \Delta$ is an orbifold.
- $\mathscr{X}$ is an algebraic stack with finitely many cyclic stabilizers.
- $\mathscr{X}$ has a canonical linear system $\omega_{\mathscr{X}}$.
- $H^0(\mathscr{X}, \omega_{\mathscr{X}}^{\otimes d}) = S_{2d}(\Gamma)$ and $H^0(\mathscr{X}, \omega_{\mathscr{X}}(\log \Delta)^{\otimes d}) = M_{2d}(\Gamma)$.

Example (The modular curve)

For $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $\Delta = \infty$ and $\mathscr{X} = X(1)$ we get

$$R(\mathscr{X}, \omega_{\mathscr{X}}(\log \Delta)) = \mathbb{C}[E_4, E_6].$$

Theorem (Voight–Zureick-Brown, 2022)

$R(\mathscr{X}, \omega_{\mathscr{X}}(\log \Delta))$ is generated by elements of degree at most $3e$ with relations of degree at most $6e$, where e is the maximal order of a stabilizer.

Stacky Surfaces

Conjecture

If $\mathscr{X}$ is a stacky surface with $\omega_{\mathscr{X}}(\log \Delta)$ inducing an embedding, then $R(\mathscr{X}, \Delta)$ is generated in degree at most $5e$ with relations in degree at most $10e$.

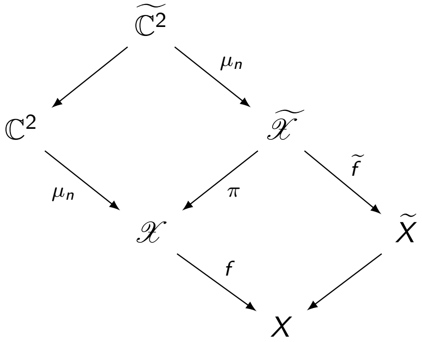
Better bounds under some technical conditions.

Other than some special cases, expect degrees to depend only on geometric invariants.

Start with X a variety, and proceed inductively. (Voight, Zureick-Brown 2022)

But the coarse space of $\mathscr{X}$ is singular!

Handling codim 2 stacky points



- $X = \mathbb{C}^2/\mu_n$
- $\widetilde{X}$ is a toric resolution of singularities
- $\mathscr{X} = [\mathbb{C}^2/\mu_n]$ has stacky structure only above the origin
- $\widetilde{\mathscr{X}}$ is root stack over the resolution cycle in $\widetilde{X}$

Relation to $\mathbb{Q}$-divisors

Assume P is a stacky point of type $(n; a, b)$, let

$$L = \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \lambda_1 a + \lambda_2 b \equiv 0 \bmod n\}$$

and let $\frac{1}{n}(l_i, m_i)$ be the boundary points of the positive cone in $L^\vee$.

Theorem (A., Chidambaram, Frengley, Schiavone, Webb)

There exist exceptional divisors E_i in $\widetilde{\mathscr{X}}$, pointwise fixed by μ_n such that

$$\pi^* K_{\mathscr{X}} = f^* K_{\widetilde{X}} + \sum_i (n - l_i - m_i) E_i.$$

Previous work

- Conjecture holds for X an algebraic surface ($e = 1$) with $\Delta = 0$. (Ciliberto 1983, Reid 1988)
- Section rings of $\mathbb{Q}$-divisors on minimal rational surfaces (Landesman, Ruhm, Zhang 2018) are related via birational maps, using the theorem.
- Spin canonical rings of log stacky curves (Landesman, Ruhm, Zhang 2016) are related via the hyperplane section principle.

Base case - log canonical rings

Theorem (A., Chidambaram, Frengley, Schiavone, Webb)

If X is regular, $\omega_X(\log \Delta)$ is ample and $p_a(\Delta) = 1$, then $R(X, \Delta)$ is generated in degree at most 5 with relations in degree at most 10.

Tools - Riemann-Roch, adjunction and Kodaira vanishing due to ampleness of $\omega_X(\log \Delta)$.

- Under some technical assumptions, can get better bounds.
- In many cases, obtain explicit Gröbner bases.

Induction step

Birational map $\mathscr{X} \to \mathscr{X}'$ where

$$Q \in \mathscr{X} \mapsto P' \in \mathscr{X}'$$

with degree $e \geq 2$.

$R = R(\mathscr{X}, \Delta)$ is an R'-algebra for $R' = R(\mathscr{X}', \Delta)$

Goal

Explicit description of generators and relations for R' over R.

Proposition

Assume Q is of type $(e; 1, 1)$. For $3 \leq i \leq e$, a general choice of $y_i \in H^0(\mathscr{X}, \omega_{\mathscr{X}}(\log \Delta)^{\otimes i})$ minimally generates R as an R'-algebra.

Issue - $\omega_{\mathscr{X}'}(\log \Delta)$ might not be ample.

Application - Hilbert modular surfaces

Let F be a real quadratic field. Consider the moduli space $\mathscr{Y}_F$ of abelian surfaces A with RM by $\mathbb{Z}_F$.

Theorem (Rapoport, 1978)

$\mathscr{Y}_F$ admits a compactification $\mathscr{X}_F$ with boundary Δ such that $(\mathscr{X}, \Delta)$ is a log stacky surface. $R(\mathscr{X}, \Delta)$ is the graded ring of Hilbert modular forms of parallel even weight.

Moreover, we have $p_a(\Delta) = 1$ and

Theorem (Baily-Borel, 1966)

In the above setting, $\omega_{\mathscr{X}}(\log \Delta)$ is ample.

Base case - Hilbert modular surfaces

Example

Let $F = \mathbb{Q}(\sqrt{5})$, and consider the moduli space $Y_{F,2}$ of abelian surfaces A with RM by $\mathbb{Z}_F$ and a basis for $A[2]$. Then $X_{F,2}$ is a (non-stacky) surface.
One computes that $h^0(\Delta) = 5$, $(K_X + \Delta)^2 = 8$ and $\chi = 1$, so

$$\Phi(R; t) = \frac{1 + 2t + 2t^2 + 2t^3 + t^4}{(1-t)^3} = \frac{1 - t^2 - t^4 + t^6}{(1-t)^5},$$

corresponding to

$$R(X, \Delta) = k[x_1, \dots, x_5]/(f_2, f_4).$$

This matches an explicit description (van der Geer, 1980).